

THE WEDDERBURN B-DECOMPOSITION FOR ALTERNATIVE BARIC ALGEBRAS

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Abstract

In this paper we deal with the Wedderburn b -decomposition for
alternative baric algebras.

1 Baric algebras

Baric algebras play a central role in the theory of genetic algebras. They were introduced by I. M. H. Etherington, in [1], in order to give an algebraic treatment to Genetic Populations. Several classes of baric algebras have been defined, such as: train, Bernstein, special triangular, etc.

In this paper F is a field of characteristic $\neq 2, 3, 5$. Let U be an algebra over F not necessarily associative, commutative or finite dimensional. If $\omega : U \longrightarrow F$ is a nonzero homomorphism of algebras, then the ordered pair (U, ω) will be called a *baric algebra* or *b-algebra* over F and ω its *weight function* or simply its *weight*. For $x \in U$, $\omega(x)$ is called *weight* of x .

When B is a subalgebra of U and $B \not\subseteq \ker \omega$, then B is called a *b-subalgebra* of (U, ω) . In this case, (B, ω_B) is a b-algebra, where $\omega_B = \omega|_B : B \longrightarrow F$. The subset $\text{bar}(B) = \{x \in B \mid \omega(x) = 0\}$ is a two-side ideal of B of codimension 1, called *bar ideal* of B . For all $b \in B$ with $\omega(b) \neq 0$, we have $B = Fb \oplus \text{bar}(B)$. If $\text{bar}(B)$ is a two-side ideal of $\text{bar}(U)$ (then by [2,

Proposition 1.1], it is also a two-sided ideal of U), then B is called *normal b-subalgebra* of (U, ω) . If $I \subseteq \text{bar}(B)$ is a two-side ideal of B , then I is called *b-ideal* of B .

Let (U, ω) be a b-algebra. A subset B is called *maximal (normal) b-subalgebra* of U if B is a (normal) b-subalgebra of U and there is no (normal) b-subalgebra C of U such that $B \subset C \subset U$. A subset I is called *maximal b-ideal* of U if I is a b-ideal of U , $I \neq \text{bar}(U)$ and there is no b-ideal J of U such that $I \subset J \subset \text{bar}(U)$.

A nonzero element $e \in U$ is called an *idempotent* if $e^2 = e$ and *nontrivial idempotent* if it is an idempotent different from multiplicative identity element. If (U, ω) is a b-algebra and $e \in U$ is an idempotent, then $\omega(e) = 0$ or $\omega(e) = 1$. When $\omega(e) = 1$, then e is called *idempotent of weight 1*.

A b-algebra (U, ω) is called *b-simple* if for all normal b-subalgebra B of U , $\text{bar}(B) = (0)$ or $\text{bar}(B) = \text{bar}(U)$. When (U, ω) has an idempotent of weight 1, then (U, ω) is b-simple if, and only if, its only b-ideals are (0) and $\text{bar}(U)$.

Let (U, ω) be a b-algebra. We define the *bar-radical* or *b-radical* of U , denoted by $\text{rad}(U)$, as: $\text{rad}(U) = (0)$, if (U, ω) is b-simple, otherwise as $\text{rad}(U) = \bigcap \text{bar}(B)$, where B runs over the maximal normal b-subalgebra of U . Of course, $\text{rad}(U)$ is a b-ideal of U .

We say that U is *b-semisimple* if $\text{rad}(U) = (0)$.

2 Alternative algebras

In this section, we present some definitions and properties of alternative algebras and prove some results which will be used later.

An algebra U over a field F is called *alternative algebra* if it satisfies the identities:

$$(x, x, y) = (y, x, x) = 0, \quad (1)$$

for all $x, y \in U$, where the $(x, y, z) = (xy)z - x(yz)$ is the *associator* of the elements x, y, z .

Let U be an alternative algebra over F . Then, U is a power-associative algebra and if U has an idempotent e , then U is the vector space direct sum $U = U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00}$, where

$$U_{ij} = \{x_{ij} \in U \mid ex_{ij} = ix_{ij} \text{ and } x_{ij}e = jx_{ij}\} \quad (i, j = 0, 1)$$

satisfying the multiplicative relations $U_{ij}U_{jl} \subset U_{il}$, $U_{ij}U_{ij} \subset U_{ji}$ and $U_{ij}U_{kl} = 0$, $j \neq k$, $(i, j, l = 0, 1)$, see [2].

A set of idempotents $\{e_1, \dots, e_t\}$, in an (arbitrary) alternative algebra, is called *pairwise orthogonal* in case $e_i e_j = 0$ for $i \neq j$. Note that any sum $e = e_1 + \dots + e_t$, of pairwise orthogonal idempotents ($t \geq 1$), is an idempotent. Also, $ee_i = e_i e = e_i$, ($i = 1, \dots, t$).

A more refined Peirce decomposition for an alternative algebra than the one given above is the following decomposition relative to a set $\{e_1, \dots, e_t\}$, of pairwise orthogonal idempotents in U : U is the vector space direct sum

$$U = \bigoplus_{i,j} U_{ij} \quad (i, j = 0, 1, \dots, t),$$

where $U_{ij} = \{x_{ij} \in U \mid e_k x_{ij} = \delta_{ki} x_{ij} \text{ and } x_{ij} e_k = \delta_{jk} x_{ij} \text{ for } (k = 1, \dots, t)\}$ ($i, j = 0, 1, \dots, t$), satisfying the multiplicative relations:

$$U_{ij} U_{jl} \subset U_{il} \quad (i, j, l = 0, 1, \dots, t), \quad (2)$$

$$U_{ij} U_{ij} \subset U_{ji} \quad (i, j = 0, 1, \dots, t), \quad (3)$$

$$U_{ij} U_{kl} = 0 \quad j \neq k, \quad (i, j) \neq (k, l) \quad (i, j, k, l = 0, 1, \dots, t), \quad (4)$$

where δ_{jk} ($j, k = 0, 1, \dots, t$) is the *Kronecker delta*.

An nonzero ideal I of an alternative algebra U is called *minimal* if for any ideal of U such that $(0) \subset J \subset I$, then $J = (0)$ or $J = I$.

Let U be a finite dimensional alternative algebra over F , since U is a power-associative algebra, then by [2] U has a unique maximal nilideal, we define *nilradical* $R(U)$ of U as the maximal nil ideal of U . Let us say that U is *simple* when its only ideals are the trivial ideals and U is not a zero algebra. If $R(U) = 0$, then U is called *semisimple*.

Lemma 2.1. *Let U be a finite dimensional alternative algebra over F with a non trivial idempotent e . If $U = \bigoplus_{i,j} U_{ij}$ ($i, j = 0, 1$), relative to e , then*

$$R(U_{ii}) = R(U) \cap U_{ii} \quad (i = 0, 1).$$

Proof. See [2, Corollary 3.8] . □

Proposition 2.1. *Let U be a finite dimensional alternative algebra. If I is a minimal ideal of U , then either $I^2 = 0$ or I is simple.*

Proof. [3, Chap. VIII, Theorem 10]. □

3 Baric alternative algebra

In this section, we introduce a notion of Wedderburn b-decomposition of a b -alternative algebra and we present conditions for which it has such decomposition.

If (U, ω) is a b -algebra and I is a b -ideal of U , then $(U/I, \bar{\omega})$ is a b -algebra, where $\bar{\omega}(u + I) = \omega(u)$.

Definition 3.1. Let (U, ω) be b -alternative algebra over a field F . We say that U has a Wedderburn b -decomposition if we can decompose U as a direct sum $U = S \oplus V \oplus \text{rad}(U)$ (vector space direct sum), where S is a b -semisimple b -subalgebra of U and V is a vector subspace of $\text{bar}(U)$ such that $V^2 \subset \text{rad}(U)$.

Lemma 3.1. Let U be a finite dimensional b -alternative algebra over F with unity element 1 and I a b -ideal of U such that $I \subset R(U)$. If $\overline{u_1}$ is a nonzero idempotent of $\text{bar}(U/I)$, then there is an idempotent e_1 in $\text{bar}(U)$ such that $\overline{e_1} = \overline{u_1}$. Moreover, if $\text{bar}(U)$ is an algebra with a unity f and $\overline{f} = \overline{u_1}$, then $f = e_1$.

Proof. Let us consider the quotient algebra $U/I = \{\overline{x} \mid x \in U\}$ and the application $\overline{\omega} : U/I \rightarrow F$ defined by $\overline{\omega}(\overline{x}) = \omega(x)$, for all $x \in U$. Then $\overline{\omega}$ is a nonzero algebra homomorphism and therefore $(U/I, \overline{\omega})$ is a b -algebra such that $U/I = F\overline{1} \oplus \text{bar}(U/I)$, where $\text{bar}(U/I) = \text{bar}(U)/I$.

Next, since $\overline{u_1}$ is an idempotent of $\text{bar}(U/I)$, then any representative u_1 of $\overline{u_1}$ is non nilpotent and belongs to $\text{bar}(U)$. It follows that the subalgebra generated by the element u_1 is a non nil subalgebra of $\text{bar}(U)$. This implies that $\text{bar}(U)$ has an idempotent $e_1 = \sum_i \alpha_i u_1^i$, $\alpha_i \in F$, verifying $\overline{e_1} = \alpha \overline{u_1}$, $\alpha \in F$, $\alpha = \sum_i \alpha_i$. Since $e_1 \notin R(U)$, then $e_1 \notin I$ and it follows that $\overline{e_1} \neq \overline{0}$ and $\overline{e_1} = \alpha \overline{u_1}$. Hence $\alpha = 1$ and $\overline{e_1} = \overline{u_1}$. Moreover, if $\text{bar}(U)$ is an algebra with a multiplicative unity f and $\overline{f} = \overline{u_1}$, then $\overline{f} = \overline{e_1}$ which implies $f - e_1 \in I$. Since $(f - e_1)^2 = f - e_1$, then $f = e_1$. \square

Lemma 3.2. Let U be a finite dimensional b -alternative algebra over F with unity element 1 and J a b -ideal of U such that $J \subset R(U)$. If $\{\overline{u_1}, \dots, \overline{u_t}\}$ is a set of nonzero pairwise orthogonal idempotents of $\text{bar}(U/J)$, then there are a set of nonzero pairwise orthogonal idempotents $\{e_1, \dots, e_t\}$ of $\text{bar}(U)$ verifying $\overline{e_i} = \overline{u_i}$ ($i = 1, \dots, t$). Moreover, if e is any idempotent of $\text{bar}(U)$ such that $\overline{e} = \sum_{i=1}^t \overline{u_i}$, we may choose by e_i such that $e = \sum_{i=1}^t e_i$.

Proof. To prove this lemma we use the principle of mathematical induction. For $t = 1$, the result is true, by Lemma 3.1. Now, suppose that for a positive integer $t \geq 1$, the lemma is true. Then for the set of nonzero pairwise orthogonal idempotents $\{\overline{u_1}, \dots, \overline{u_{t+1}}\}$ of $\text{bar}(U/J)$, there is a set of nonzero pairwise orthogonal idempotents $\{e_1, \dots, e_t\}$, of $\text{bar}(U)$, verifying $\overline{e_i} = \overline{u_i}$ ($i = 1, \dots, t$), by the principle of mathematical induction. Let us consider the Peirce decompositions:

$$\text{bar}(U) = \bigoplus_{i,j} \text{bar}(U)_{ij} \text{ and } \text{bar}(U/J) = \bigoplus_{i,j} \text{bar}(U/J)_{ij} \text{ (} i, j = 0, 1 \text{),}$$

relative to idempotents $e = \sum_{i=1}^t e_i$ and $\bar{e} = \sum_{i=1}^t \bar{e}_i$, respectively. It follows that:

- (i) $e_i \in \text{bar}(U)_{11}$ ($i = 1, \dots, t$);
- (ii) $\bar{e}_i \in \text{bar}(U/J)_{11}$ ($i = 1, \dots, t$);
- (iii) $\overline{u_{t+1}} \in \text{bar}(U/J)_{00}$.

Let us define $P = F1 \oplus \text{bar}(U)_{00}$. Then (P, ω_P) is a finite dimensional b -subalgebra of (U, ω) with unity element 1 and $\text{bar}(P) = \text{bar}(U)_{00}$, where $\omega_P := \omega|_P$. Let us define $K = J \cap \text{bar}(U)_{00}$. It is easy to check that K is a b -ideal of P such that $K \subset R(P)$, because $R(P) = R(\text{bar}(P)) = R(\text{bar}(U)_{00})$, by [6, Proposition 4.1] and the fact that $R(\text{bar}(U)_{00}) = R(\text{bar}(U)) \cap \text{bar}(U)_{00} = R(U) \cap \text{bar}(U)_{00}$, according to Lemma 2.1, where $R(\text{bar}(U)) = R(U)$, again by [6, Proposition 4.1].

Let us consider the quotient algebra $P/K = \{\tilde{x} \mid x \in P\}$ and $\tilde{\omega} : P/K \rightarrow F$, defined by $\tilde{\omega}_P(\tilde{x}) := \omega_P(x)$ for all $x \in P$. Certainly, $(P/K, \tilde{\omega}_P)$ is a b -algebra such that $\text{bar}(P/K) = \text{bar}(P)/K = \text{bar}(U)_{00}/(J \cap \text{bar}(U)_{00})$.

Also observe that

$$\text{bar}(U/J)_{00} = (\text{bar}(U)_{00} + J)/J \cong \text{bar}(U)_{00}/(J \cap \text{bar}(U)_{00}). \quad (5)$$

Since $\overline{u_{t+1}} \in \text{bar}(U/J)_{00}$, we can assume that $u_{t+1} \in \text{bar}(U)_{00}$. In fact, let us write $u_{t+1} = a_{11} + a_{10} + a_{01} + a_{00}$, where $a_{ij} \in \text{bar}(U)_{ij}$ ($i, j = 0, 1$). Then $\overline{u_{t+1}} = \overline{a_{11}} + \overline{a_{10}} + \overline{a_{01}} + \overline{a_{00}}$ which implies $\bar{0} = \bar{e} \overline{u_{t+1}} = \overline{a_{11}} + \overline{a_{10}}$ and $\bar{0} = \overline{u_{t+1}} \bar{e} = \overline{a_{11}} + \overline{a_{01}}$. Thus, $\overline{a_{11}} = \overline{a_{10}} = \overline{a_{01}} = \bar{0}$ implying $\overline{u_{t+1}} = \overline{a_{00}}$.

From the isomorphism, in (5), and the assumption on the element idempotent $\overline{u_{t+1}}$, we have $\widetilde{u_{t+1}} \in \text{bar}(P/K)$ which implies that there is an idempotent $e_{t+1} \in \text{bar}(P)$ such that $\widetilde{e_{t+1}} = \widetilde{u_{t+1}}$, by Lemma 3.1. From the isomorphism, in (5), we conclude that $e_{t+1} \in \text{bar}(U)$ and $\overline{e_{t+1}} = \overline{u_{t+1}}$. Since $e_{t+1} \in \text{bar}(U)_{00}$, then the elements idempotent e_1, \dots, e_t, e_{t+1} are pairwise orthogonal.

Finally, suppose that e is an arbitrary idempotent of $\text{bar}(U)$ such that $\bar{e} = \sum_{i=1}^t \bar{u}_i$. Let us consider the Peirce decompositions $\text{bar}(U) = \bigoplus_{i,j} \text{bar}(U)_{ij}$ and $\text{bar}(U/J) = \bigoplus_{i,j} \text{bar}(U/J)_{ij}$ ($i, j = 0, 1$), relative to idempotents e and \bar{e} , respectively. It follows that $\bar{u}_i \in \text{bar}(U/J)_{11}$ ($i = 1, \dots, t$).

Let us define the vector subspace of $Q = F1 \oplus \text{bar}(U)_{11}$ of U . Naturally, Q is an subalgebra of U such that $Q \not\subset \ker(\omega)$. It follows that (Q, ω_Q) , where $\omega_Q := \omega|_Q$, is a finite dimensional b -subalgebra of U with unity element 1 and $\text{bar}(Q) = \text{bar}(U)_{11}$. Let us define $L = J \cap \text{bar}(U)_{11}$. As in the

previous definitions, certainly L is a b -ideal of Q such that $L \subset R(Q)$, because $R(Q) = R(\text{bar}(Q)) = R(\text{bar}(U)_{11})$, by [6, Proposition 4.1] and the fact that $R(\text{bar}(U)_{11}) = R(\text{bar}(U)) \cap \text{bar}(U)_{11} = R(U) \cap \text{bar}(U)_{11}$, according to Lemma 2.1, where $R(\text{bar}(U)) = R(U)$, again by [6, Proposition 4.1].

Let us take the quotient algebra $Q/L = \{\tilde{x} \mid x \in Q\}$ and $\tilde{\omega} : Q/L \rightarrow F$, defined by $\tilde{\omega}_Q(\tilde{x}) := \omega_Q(x)$ for all $x \in Q$. Again, we have that $(Q/L, \tilde{\omega}_Q)$ is a b -algebra such that $\text{bar}(Q/L) = \text{bar}(Q)/L = \text{bar}(U)_{11}/(J \cap \text{bar}(U)_{11})$.

Now, let us observe that

$$\text{bar}(U/J)_{11} = (\text{bar}(U)_{11} + J)/J \cong \text{bar}(U)_{11}/(J \cap \text{bar}(U)_{11}). \quad (6)$$

Since $\overline{u_i} \in \text{bar}(U/J)_{11}$ and $\overline{u_i} = \overline{e} \overline{u_i} \overline{e}$ ($i = 1, \dots, t$), we can take a representative u_i of $\overline{u_i}$ in $\text{bar}(U)_{11}$ ($i = 1, \dots, t$).

From the isomorphism, in(6), and the assumption on the element idempotent $\overline{u_i}$, we have that $\tilde{u_i} \in \text{bar}(Q/L)$ which implies that there is a set of idempotents e_1, \dots, e_t , in $\text{bar}(Q)$, pairwise orthogonal, such that $\tilde{e_i} = \tilde{u_i}$ ($i = 1, \dots, t$).

As the idempotent e is a multiplicative unity in the subalgebra $\text{bar}(U)_{11}$ and $\overline{e} = \sum_{i=1}^t \overline{e_i}$, then $e - \sum_{i=1}^t e_i \in J \subset R(U)$ and $(e - \sum_{i=1}^t e_i)^2 = e - \sum_{i=1}^t e_i$ which implies $e = \sum_{i=1}^t e_i$. \square

Lemma 3.3. *Let (U, ω) be a finite dimensional b -algebra of (γ, δ) type with unity element 1 and J a b -ideal of U such that $J \subset R(U)$. If $\text{bar}(U/J)$ contains a total matrix algebra \mathfrak{M}_t of degree t with identity element \overline{u} and f is an idempotent of $\text{bar}(U)$ such that $\overline{f} = \overline{u}$, then $\text{bar}(U)$ contains a total matrix algebra \mathfrak{M} of degree t with identity element f such that $\overline{\mathfrak{M}} = \mathfrak{M}_t$.*

Proof. Let \mathfrak{M}_t be a total matrix algebra \mathfrak{M}_t of degree t with identity element \overline{u} . By hypothesis we have $\mathfrak{M}_t = \{\overline{u_{ij}} \mid i, j = 1, \dots, t\}$, with the familiar multiplication table $\overline{u_{ij}} \overline{u_{kl}} = \delta_{jk} \overline{u_{il}}$ ($i, j = 1, \dots, t$).

By Lemma 3.2, there exist pairwise orthogonal idempotents f_{11}, \dots, f_{tt} , in $\text{bar}(U)$, such that $\overline{f_{ii}} = \overline{u_{ii}}$ ($i = 1, \dots, t$) and $f = \sum_{i=1}^t f_{ii}$.

Let us consider the Peirce decompositions $\text{bar}(U) = \bigoplus_{i,j} \text{bar}(U)_{ij}$ and $\text{bar}(U/J) = \bigoplus_{i,j} \text{bar}(U/J)_{ij}$ ($i, j = 0, 1, \dots, t$), relative to the sets of idempotents $\{f_{11}, \dots, f_{tt}\}$ and $\{\overline{f_{11}}, \dots, \overline{f_{tt}}\}$, respectively. It follows that: (i) $f_{ii} \in \text{bar}(U)_{ii}$ ($i = 1, \dots, t$); and (ii) $\overline{f_{ii}} \in \text{bar}(U/J)_{ii}$ ($i = 1, \dots, t$).

Now, let us observe that for every index $i = 2, \dots, t$, we can take the representative u_{i1} , of $\overline{u_{i1}} \in \mathfrak{M}_t$, in $\text{bar}(U)_{i1}$. For a $i = 1$, let us take $u_{11} = f_{11}$. Similarly, for every index $j = 2, \dots, t$, we can take the representative u_{1j} , of $\overline{u_{1j}} \in \mathfrak{M}$, in $\text{bar}(U)_{1j}$.

Yet, since $\overline{u_{1j}u_{j1}} = \overline{f_{11}}$ ($j = 1, \dots, t$), then $u_{1j}u_{j1} = f_{11} + a_j$, where $a_j \in J \cap \text{bar}(U)_{11}$ is a nilpotent element. Let us consider m the smallest positive integer such that $a_j^m = 0$ and let us define $b_j = \sum_{i=1}^{m-1} (-a_j)^i$. Then: (i) $b_j \in J \cap \text{bar}(U)_{11}$; (ii) $b_j a_j = -\sum_{i=2}^{m-1} (-a_j)^i$; and (iii) $a_j + b_j + b_j a_j = 0$. It follows that $(f_{11} + b_j)(f_{11} + a_j) = f_{11} + a_j + b_j + b_j a_j = f_{11}$.

Let us define $f_{i1} = u_{i1}$ and $f_{1j} = (f_{11} + b_j)u_{1j}$ ($i, j = 2, \dots, t$). Then $f_{1j}f_{j1} = ((f_{11} + b_j)u_{1j})u_{j1} = (f_{11}u_{1j})u_{j1} + (b_j u_{1j})u_{j1} = f_{11}(u_{1j}u_{j1}) + b_j(u_{1j}u_{j1}) = f_{11}(f_{11} + a_j) + b_j(f_{11} + a_j) = f_{11}$. Next, let us define $f_{ij} = f_{i1}f_{1j}$ ($i \neq j; i, j = 2, \dots, t$). From a direct calculus, we have $\overline{f_{ij}} = \overline{u_{ij}}$ and $f_{ij}f_{kl} = \delta_{jk}f_{il}$ ($i, j, k, l = 1, \dots, t$). Thus, the set $\{f_{ij} \mid i, j = 1, \dots, t\}$ is a basis for a total matrix algebra \mathfrak{M} of degree t , in $\text{bar}(U)$, with identity element f such that $\overline{\mathfrak{M}} = \mathfrak{M}_t$. \square

Lemma 3.4. *Let (U, ω) be a finite dimensional b -alternative algebra with unity element 1 and J a b -ideal of U such that $J \subset R(U)$. If $\text{bar}(U/J)$ contains a direct sum of b -ideals $\mathfrak{M}_{t_1} \oplus \dots \oplus \mathfrak{M}_{t_s}$, where each \mathfrak{M}_{t_i} is a total matrix algebra of degree t_i ($i = 1, \dots, s$), then $\text{bar}(U)$ contains a direct sum of pairwise orthogonal subalgebras $\mathfrak{M}_1 \oplus \dots \oplus \mathfrak{M}_s$, where each \mathfrak{M}_i is a total matrix algebra of degree t_i ($i = 1, \dots, s$), such that $\overline{\mathfrak{M}_i} = \mathfrak{M}_{t_i}$ and*

$$\mathfrak{M}_1 \oplus \dots \oplus \mathfrak{M}_s \cong \mathfrak{M}_{t_1} \oplus \dots \oplus \mathfrak{M}_{t_s}.$$

Proof. Let $\overline{e_{t_i}}$ be the unity element of \mathfrak{M}_{t_i} ($i = 1, \dots, s$). By Lemma 3.2, $\text{bar}(U)$ has a set of idempotents e_1, \dots, e_s , pairwise orthogonal, such that $\overline{e_i} = \overline{e_{t_i}}$ ($i = 1, \dots, s$). This implies that $\text{bar}(U)$ contains a total matrix algebra \mathfrak{M}_i of degree t_i with identity element e_i such that $\overline{\mathfrak{M}_i} = \mathfrak{M}_{t_i}$, by Lemma 3.3.

Let us consider the Peirce decomposition $\text{bar}(U) = \bigoplus_{i,j} \text{bar}(U)_{ij}$ ($i, j = 1, \dots, s$), relative to set of idempotents $\{e_1, \dots, e_s\}$. For all element $x_i \in \mathfrak{M}_i$ ($i = 1, \dots, s$), we have $x_i = e_i x_i$. But in an alternative algebra each associator $(x, e_j, e_l) = 0$ and $(e_j, e_l, x) = 0$ ($j, l = 1, \dots, s$), which implies $e_k x_i = e_k(e_i x_i) = (e_k e_i) x_i = \delta_{ki} x_i$. Similarly, we show $x_i e_k = \delta_{ik} x_i$. Thus, $\mathfrak{M}_i \subset \text{bar}(U)_{ii}$ ($i = 1, \dots, s$). Since the subalgebras $\text{bar}(U)_{ii}$ ($i = 1, \dots, s$) are pairwise orthogonal, then the sum $\mathfrak{M}_1 \oplus \dots \oplus \mathfrak{M}_s$, is a direct sum, pairwise orthogonal, such that $\mathfrak{M}_1 \oplus \dots \oplus \mathfrak{M}_s \cong \mathfrak{M}_{t_1} \oplus \dots \oplus \mathfrak{M}_{t_s}$. \square

Lemma 3.5. *Let (U, ω) be a finite dimensional b -alternative algebra with unity element 1 and J a b -ideal of U such that $J \subset R(U)$. If $\text{bar}(U/J)$ contains a direct sum of b -ideals $J_1 \oplus \dots \oplus J_r$ such that $J_i^2 = \overline{0}$ ($i = 1, \dots, r$), then $\text{bar}(U)$ contains a vector subspace V such that $V \cong \overline{V} = J_1 \oplus \dots \oplus J_r$ and $V^2 \subset \text{rad}(U)$.*

Proof. Let $\{\overline{v_{i,1}}, \overline{v_{i,2}}, \dots, \overline{v_{i,r_i}}\}$ be a basis of vector subspace J_i ($i = 1, \dots, r$) and $v_{i,j}$ a representative of the class $\overline{v_{i,j}}$ ($i = 1, \dots, r$; $j = 1, \dots, r_i$), in $\text{bar}(U)$. From a direct calculus, we have: (i) the set $\bigcup_{i=1}^r \{v_{i,1}, v_{i,2}, \dots, v_{i,r_i}\}$ is linearly independent; and (ii) $v_{i,j}v_{k,l} \in \text{rad}(U)$ ($i, k = 1, \dots, r$) and ($j = 1, \dots, r_i$; $l = 1, \dots, r_k$).

Let us define V the vector subspace generated by the set

$$\bigcup_{i=1}^r \{v_{i,1}, v_{i,2}, \dots, v_{i,r_i}\}.$$

It follows that $V \cong \overline{V} = J_1 \oplus \dots \oplus J_r$ and $V^2 \subset \text{rad}(U)$. \square

Lemma 3.6. *Let (U, ω) be a finite dimensional b -alternative algebra with unity element 1 and J a b -ideal of U such that $J^2 = 0$. If $\text{bar}(U/J)$ contains a direct sum of b -ideals $I_1 \oplus \dots \oplus I_r$ such that I_i is a split Cayley algebra ($i = 1, \dots, r$), then $\text{bar}(U)$ contains a subalgebra $\mathcal{C} \cong I_1 \oplus \dots \oplus I_r$.*

Proof. We may take $I_k = F_2 + \overline{w_k}F_2$, $\overline{w_k}^2 = \overline{1}$ by [2, Lemma 3.16] where F_2 is the algebra of all 2×2 matrices over F , $k = 1, \dots, r$. By Lemma 3.4, $\text{bar}(U)$ contains a total matrix algebra $\mathcal{D} \cong F_2$ such that \mathcal{D} contains an identity element and the matrix basis $\{e_{ij}\}$ of \mathcal{D} yields the matrix basis $\{\overline{e_{ij}}\}$ of F_2 . Let $\iota : \mathcal{D} \rightarrow \mathcal{D}$ the involution in \mathcal{D} . We know $x + \iota(x) = t(x)1$ for all $x \in \mathcal{D}$, where $t(x)$ is the trace of x and 1 is the identity element of \mathcal{D} by [2, page 45]. Note that $\iota(\overline{x}) = \overline{\iota(x)}$, $a(\overline{w_k}b) = \overline{w_k}(\iota(a)b)$, $(\overline{w_k}a)b = \overline{w_k}(ba)$ and $(\overline{w_k}a)(\overline{w_k}b) = b\iota(a)$ for $x, a, b \in \mathcal{D}$, we have indicated to the reader [2, Chapter III, Sec. 4] for Cayley Algebras. In order to prove the lemma, it is sufficient to show the existence of $v_k \notin \mathcal{D}$ satisfying $v_k^2 = 1$, $\overline{v_k} = \overline{w_k}$ and $xv_k = v_k\iota(x)$ for all $x \in \mathcal{D}$.

Consider $\overline{f_{ij}} = \overline{w_k} \overline{e_{jj}}$ for $i \neq j$ ($i, j = 1, 2$).

Using the Peirce decomposition of $\text{bar}(U)$ relative to $e_1 = e_{11}$, $e_2 = e_{22}$, we may take $f_{ij} \in \text{bar}(U)_{ij}$ ($i \neq j$). In fact $\overline{e_{ii}}(\overline{f_{ij}}\overline{e_{jj}}) = \overline{e_{ii}}(\overline{w_k} \overline{e_{jj}}^2) = \overline{w_k} \iota(e_{ii})e_{jj} = \overline{w_k} \overline{e_{jj}} = \overline{f_{ij}}$. Now $\overline{e_{ji}} \overline{f_{ij}} = \overline{e_{ji}} (\overline{w_k} \overline{e_{ij}}) = -\overline{w_k}(\overline{e_{ji}} \overline{e_{jj}}) = 0$, implying that

$$e_{ji}f_{ij} = c_j, \quad c_j \in J \cap \text{bar}(U)_{jj} \quad (i \neq j; i, j = 1, 2).$$

Write $h_{ij} = f_{ij} - e_{ij}c_j$. Then $h_{ij} \in \text{bar}(U)_{ij}$, $\overline{h_{ij}} = \overline{f_{ij}}$, and

$$e_{ji}h_{ij} = h_{ij}e_{ji} = 0 \quad (i \neq j; i, j = 1, 2).$$

In fact by Lemma 3.3 we know $e_{ji}e_{ij} = e_{jj}$, so $e_{ji}h_{ij} = c_j - e_{ji}(e_{ij}c_j) = c_j - (e_{ji}e_{ij})c_j = 0$. Also $e_{ij}c_j = e_{ij}(e_{ji}f_{ij}) = (e_{ij}e_{ji})f_{ij} - (e_{ij}e_{ji}, f_{ij}) =$

$f_{ij} + (f_{ij}, e_{ji}, e_{ij}) = f_{ij} + (f_{ij}e_{ji})e_{ij} - f_{ij} = (f_{ij}e_{ji})e_{ij}$, so that

$$h_{ij}e_{ji} = f_{ij}e_{ji} - [(f_{ij}e_{ji})e_{ij}]e_{ji} = 0.$$

Now $\overline{h_{ij}} \overline{h_{ji}} = \overline{f_{ij}} \overline{f_{ji}} = \overline{e_{ii}} \overline{\iota(e_{jj})} = \overline{e_{ii}}$ implies that

$$h_{ij}h_{ji} = e_{ii} + a_i, \quad a_i \in J \cap \text{bar}(U)_{ii} \quad (i \neq j; i, j = 1, 2).$$

Then $a_i^2 = 0$ since $J^2 = 0$, and

$$(e_{ii} - a_i)(e_{ii} + a_i) = e_{ii} = (e_{ii} + a_i)(e_{ii} - a_i) \quad (i = 1, 2).$$

Write $p_{12} = (e_{11} - a_1)h_{12}$, $p_{21} = h_{21}$. Then $p_{ij} \in \text{bar}(U)_{ij}$, $\overline{p_{ij}} = \overline{f_{ij}}$, and we shall prove

$$p_{ij}p_{ji} = e_{ii} \quad (i \neq j; i, j = 1, 2).$$

In fact,

$$p_{12}p_{21} = [(e_{11} - a_1)h_{12}]h_{21} = (e_{11} - a_1)(h_{12}h_{21}) = (e_{11} - a_1)(e_{11} + a_1) = e_{11}.$$

But

$$a_i h_{ij} = (h_{ij}h_{ji} - e_{ii})h_{ij} = h_{ij}(h_{ji}h_{ij}) - h_{ij} = h_{ij}(e_{jj} + a_j) - h_{ij} = h_{ij}a_j,$$

so that $p_{12} = h_{12} - a_1h_{12} = h_{12} - h_{12}a_2 = h_{12}(e_{22} - a_2)$ and $p_{21}p_{12} = h_{21}[h_{12}(e_{22} - a_2)] = (h_{21}h_{12})(e_{22} - a_2) = e_{22}$. Also we have $e_{ij}p_{ji} = \overline{p_{ji}}e_{ij} = 0$ ($i \neq j; i, j = 1, 2$). Thus write $v_k = p_{12} + p_{21}$. Then $\overline{v_k} = \overline{f_{12}} + \overline{f_{21}} = \overline{w_k}$, implying $v_k \notin \mathcal{D}$. Also

$$v_k^2 = (p_{12} + p_{21})^2 = e_{11} + e_{22} = 1.$$

Writing $b = \alpha_1 e_{11} + \alpha_2 e_{12} + \alpha_3 e_{21} + \alpha_4 e_{22}$, we have $\iota(b) = \alpha_4 e_{11} - \alpha_2 e_{12} - \alpha_3 e_{21} + \alpha_1 e_{22}$,

$$\begin{aligned} bv_k &= \alpha_1 p_{12} + \alpha_2 e_{12} p_{12} + \alpha_3 e_{21} p_{21} + \alpha_4 p_{21} \\ &= \alpha_1 p_{12} - \alpha_2 p_{12} e_{12} - \alpha_3 p_{21} e_{21} + \alpha_4 p_{21} = v_k \iota(b) \end{aligned}$$

which completes the proof of the lemma. \square

Theorem 3.1. *Let F be an algebraically closed field and (U, ω) be a finite dimensional b -alternative algebra over F with unity element 1. Then U has a Wedderburn b -decomposition $U = S \oplus V \oplus \text{rad}(U)$. Furthermore, $\text{bar}(S)$ is a semisimple algebra and $V \oplus \text{rad}(U)$ is a nil ideal of $\text{bar}(U)$.*

Proof. The same inductive argument based on the dimension of U which is used for associative algebras suffices to reduce the proof of the theorem to the case $\text{rad}(U)^2 = 0$. Let us take the quotient b -algebra $U/\text{rad}(U)$. By [5, Corollary 3.1], we have $\text{rad}(U/\text{rad}(U)) = 0$ which implies that $U/\text{rad}(U)$ is b -semisimple, by [5, Theorem 4.2]. So, $\text{bar}(U/\text{rad}(U))$ is a sum of minimal b -ideals $I_1 \oplus \cdots \oplus I_k \oplus I_{k+1} \oplus \cdots \oplus I_s \oplus J_{s+1} \oplus \cdots \oplus J_r$, of $U/\text{rad}(U)$, where I_i are simple associative algebras ($i = 1, \dots, k$), I_i are Cayley algebras ($i = k+1, \dots, s$) and $J_j^2 = 0$ ($s+1 \leq j \leq r$), by Proposition 2.1 and [3, Corollary 1, page 151]. Since every ideal I_i ($i = 1, \dots, k$) is a total matrix algebra \mathfrak{M}_{t_i} of degree t_i ($i = 1, \dots, k$), by [4, Corollary b, §3.5], then $\text{bar}(U)$ contains a direct sum of pairwise orthogonal total matrix algebras \mathfrak{M}_i of degree t_i ($i = 1, \dots, k$) such that $\mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_k \cong \mathfrak{M}_{t_1} \oplus \cdots \oplus \mathfrak{M}_{t_k}$, by Lemma 3.4. On the other hand, $\text{bar}(U)$ contains a vector subspace V such that $V \cong J_{s+1} \oplus \cdots \oplus J_r$ and $V^2 \subset \text{rad}(U)$, by Lemma 3.5. Moreover $\text{bar}(U)$ contains also a subalgebra $\mathcal{C} \cong I_1 \oplus \cdots \oplus I_k$, by Lemma 3.6.

Let us define $S = F1 \oplus \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_k \oplus \mathcal{C}$. Certainly, S is a b -subalgebra of U such that $\text{bar}(S)$ is a semisimple algebra and which yields S semisimple, by [6, Proposition 4.1.]. Hence, S is a b -semisimple, by [6, Lemma 4.1.]. Moreover, since $(\mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_k) \cap V = (0)$ and $\mathcal{C} \cap (\mathfrak{M}_1 + \cdots + \mathfrak{M}_k + V + \text{rad}(U)) = (0)$, then $U = S \oplus V \oplus \text{rad}(U)$.

Finally, let us show that the subspace $V \oplus \text{rad}(U)$ is a nil ideal of $\text{bar}(U)$. In fact, for arbitrary elements $x \in \text{bar}(U)$ and $y \in V \oplus \text{rad}(U)$, we have $\overline{x} = \sum_{i=1}^k \overline{a_i} + \sum_{i=k+1}^s \overline{a_i} + \sum_{j=s+1}^r \overline{b_j}$, where $\overline{a_i} \in I_i$ ($i = 1, \dots, s$) and $\overline{b_j} \in J_j$ ($j = s+1, \dots, r$), and $\overline{y} = \sum_{j=s+1}^r \overline{c_j}$, where $\overline{c_j} \in J_j$ ($j = s+1, \dots, r$). Hence $\overline{xy} = \overline{x}\overline{y} \in J_{s+1} \oplus \cdots \oplus J_r$ which implies $xy \in V \oplus \text{rad}(U)$. Similarly, we prove $yx \in V \oplus \text{rad}(U)$. Thus, $V \oplus \text{rad}(U)$ is an ideal of $\text{bar}(U)$. Since $y^2 \in \text{rad}(U)$, then y is a nilpotent element and therefore we can conclude that $V \oplus \text{rad}(U)$ is a nil ideal of $\text{bar}(U)$. \square

Theorem 3.2. (Main) *Let F be an algebraically closed field and (U, ω) a finite dimensional b -alternative algebra over F . Then U has a Wedderburn b -decomposition.*

Proof. Let us consider a principal idempotent e and take $U = U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00}$, the Peirce decomposition of U , relative to e . We know that: (i) U_{11} is a subalgebra with unity element e ; (ii) $U_{10} \oplus U_{01} \oplus U_{00} \subset R(U)$ and (iii) $R(U) = R(U_{11}) \oplus U_{10} \oplus U_{01} \oplus U_{00}$. Moreover, as the idempotent e is principal in U , then it has weight one. This implies that U_{11} is a b -subalgebra of U . Thus, U and U_{11} admit the decompositions $U = Fe \oplus \text{bar}(U)$ and $U_{11} = Fe \oplus \text{bar}(U_{11})$, respectively.

From Theorem 3.1, we can decompose U_{11} as a direct sum

$$U_{11} = S \oplus W_{11} \oplus \text{rad}(U_{11}),$$

where S is a b-semisimple b-subalgebra of U_{11} such that $\text{bar}(S)$ is a semisimple algebra, W_{11} is a vector subspace of $\text{bar}(U_{11})$ satisfying $W_{11}^2 \subset \text{rad}(U_{11})$ and $W_{11} \oplus \text{rad}(U_{11})$ is a nil ideal of $\text{bar}(U_{11})$. It follows that, S is a b-semisimple b-subalgebra of U , by [6, Proposition 4.1. and Lemma 4.1.].

Now, let us observe that

$$\text{rad}(U) \cap U_{11} \subset R(U) \cap U_{11} = R(U_{11}) \subset \text{bar}(U_{11}),$$

by [6, Teorema 4.1.] and Lemma 2.1, and $\text{bar}(U_{11}) = \text{bar}(S) \oplus W_{11} \oplus \text{rad}(U_{11})$. Hence, $\text{rad}(U) \cap U_{11} \subset W_{11} \oplus \text{rad}(U_{11})$, because $S \cap R(U_{11}) = \{0\}$. Let us take V_{11} an complementary subspace of $\text{rad}(U) \cap U_{11}$, in $W_{11} \oplus \text{rad}(U_{11})$. Then

$$W_{11} \oplus \text{rad}(U_{11}) = V_{11} \oplus (\text{rad}(U) \cap U_{11}).$$

Since $\text{rad}(U_{11}) = \text{bar}(U_{11})^2 \cap R(U_{11}) \subset \text{bar}(U)^2 \cap R(U) = \text{rad}(U)$, by [6, Theorem 4.2.], then $V_{11}^2 \subset \text{rad}(U)$. Thus

$$U_{11} = S \oplus V_{11} \oplus (\text{rad}(U) \cap U_{11}),$$

where $V_{11}^2 \subset \text{rad}(U)$.

Next, let us consider the complementary subspaces V_{10} , V_{01} and V_{00} , in U_{10} , U_{01} and U_{00} , respectively, such that $U_{10} = V_{10} \oplus \text{rad}(U) \cap U_{10}$, $U_{01} = V_{01} \oplus \text{rad}(U) \cap U_{01}$ and $U_{00} = V_{00} \oplus \text{rad}(U) \cap U_{00}$ and take the vector subspace $V = V_{11} \oplus V_{10} \oplus V_{01} \oplus V_{00}$. Certainly, V is a vector subspace of $\text{bar}(U)$ and

$$\begin{aligned} U &= U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00} \\ &= S \oplus V_{11} \oplus (\text{rad}(U) \cap U_{11}) \oplus U_{10} \oplus U_{01} \oplus U_{00} \\ &= S \oplus V_{11} \oplus (\text{rad}(U) \cap U_{11}) \oplus V_{10} \oplus (\text{rad}(U) \cap U_{10}) \\ &\quad \oplus V_{01} \oplus (\text{rad}(U) \cap U_{01}) \oplus V_{00} \oplus (\text{rad}(U) \cap U_{00}) \\ &= S \oplus V \oplus \text{rad}(U). \end{aligned}$$

Now, V_{11} and V_{10} are vector subspaces of $\text{bar}(U)$ and $R(U)$ respectively which implies $V_{11}V_{10} \subset (\text{bar}(U))^2$ and $V_{11}V_{10} \subset R(U)$. This yields

$$V_{11}V_{10} \subset R(U) \cap (\text{bar}(U))^2 = \text{rad}(U),$$

by [6, Teorema 4.2]. Similarly, we can show that the products $V_{10}V_{01}$, $V_{10}V_{00}$, $V_{01}V_{11}$, $V_{01}V_{10}$, $V_{00}V_{01}$ and V_{00}^2 are subsets of $\text{rad}(U)$. As all remaining products are zeros, then we can conclude that $V^2 \subset \text{rad}(U)$. \square

4 Final remarks

Importantly, the concept of heredity to b -algebras can not be extended to alternative algebras, we can see this through an example found in [7].

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